Indian Statistical Institute, Bangalore
B. Math. Second Year
Second Semester - Computer Science II

Final Exam | Duration: 3 hours | Date : May 11, 2015

Answer all the questions. *Each question carries 10 marks.* Max Marks: 50

1. Let $\boldsymbol{A}$ be an $n \times n$ with $det\boldsymbol{A} \neq 0$. State and prove conditions under which $\boldsymbol{A}$ can be decomposed uniquely as
$$\boldsymbol{A} = \boldsymbol{LU}$$
where $\boldsymbol{L}, \boldsymbol{U}$ are lower and upper triangular matrices. Now consider $\boldsymbol{A}$ to be symmetric and positive definite; what are $\boldsymbol{L}, \boldsymbol{U}$ and give an algorithm to solve
$$\boldsymbol{A}x = b$$
for this case.

2. Define a matrix norm and explain how it differs from a vector norm . Show that the vector norm
$$\|x\|_\infty = \max_i |x_i|$$
for $x \in \mathbb{R}^n$ induces the matrix norm
$$\|\boldsymbol{A}\|_\infty = \max_i \sum_j |a_{ij}|.$$

3. Let $\boldsymbol{A}$ be an $n \times n$ matrix with real entries. If $\|\boldsymbol{A}\|_\infty < 1$ show that $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ exists and $(\boldsymbol{A} + \boldsymbol{E})^{-1}$ also exists provided $\|\boldsymbol{A^{-1}E}\|_\infty < 1$. Now consider the matrix $\boldsymbol{B}$ with real entries
$$\begin{bmatrix} \epsilon & 0 & 0 & a \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ b & 0 & 0 & \epsilon \end{bmatrix}$$
and establish conditions on a, b for $\boldsymbol{B}^{-1}$ to exist for $\epsilon \neq 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function and $f'(x)$ denotes the derivative of f at $x \in (a, b)$. Define the difference quotient
$$D_h f(x) = \frac{f(x+h) - f(x)}{h}$$
for $0 < h < 1$ and $e(h)$ be the absolute error
$$e(h) = |f'(x) - D_h f(x)|.$$
Now, let $\tilde{f}(x)$ denote the floating point representation of f and consider the corresponding absolute error
$$E(h) = |\tilde{f}'(x) - D_h \tilde{f}(x)|.$$
Explain the behaviour of $e(h)$ and $E(h)$ as $h \to 0$.

5. Let $x, x_0, \cdots, x_n \in [a, b]$ and $f : [a, b] \to \mathbb{R}$ be a smooth function, write down the Lagrange interpolation polynomial $\mathcal{L}_n(x)$ that satisfies $\mathcal{L}_n(x_i) = f(x_i)$ for $i = 0, \cdots, n$. Show that

$$f(x) - \mathcal{L}_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\omega(x)$$

where c depends on x, $f^{(n+1)}$ denotes the $(n+1)$th derivative of f and

$$\omega(x) = \prod_{k=0}^{n}(x - x_k).$$

Derive an upper bound for $|f(x) - \mathcal{L}_n(x)|$ if the points are equally spaced

$$x_0 = a, \quad h = (b-a)/n, \quad x_j = a + jh, \qquad j = 1, \cdots, n.$$

[Hint: First show that $|\omega(x)| < n!\, h^{n+1}/4$.]